



# A contractive version of a Schur–Horn theorem in $\Pi_1$ factors

M. Argerami<sup>a,\*</sup>, P. Massey<sup>b,c,2</sup>

<sup>a</sup> *Department of Mathematics, University of Regina, Regina, SK, Canada*

<sup>b</sup> *Departamento de Matemática, Universidad Nacional de La Plata, Argentina*

<sup>c</sup> *Instituto Argentino de Matemática-Conicet, Argentina*

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## Abstract

We prove a contractive version of the Schur–Horn theorem for submajorization in  $\Pi_1$  factors that complements some previous results on the Schur–Horn theorem within this context. We obtain a reformulation of a conjecture of Arveson and Kadison regarding a strong version of the Schur–Horn theorem in  $\Pi_1$  factors in terms of submajorization and contractive orbits of positive operators.

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## 1. Introduction

Vector and matrix majorization theory play an important role in matrix analysis, mostly as tools in the study of general (convex) inequalities, unitarily invariant norm inequalities, geometry, and problems related with the description of the diagonals of matrix representations of a linear operator [1,2,4,12]. Some historical aspects of the theory of majorization are mentioned in [3,5]. The Schur–Horn theorem, coined in the papers [8,15], is probably the most remarkable among the many characterizations known for these notions (see the precise statement of the theorem

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\* Corresponding author.

*E-mail addresses:* [argerami@math.uregina.ca](mailto:argerami@math.uregina.ca) (M. Argerami), [massey@mate.unlp.edu.ar](mailto:massey@mate.unlp.edu.ar) (P. Massey).

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after Proposition 2.3). It is thus natural to search for analogues of this result in contexts where majorization theory has been extended [5–7,10,13]. Among these analogues let us mention the work of Neumann [13] for selfadjoint operators in  $\mathcal{B}(\mathcal{H})$ , the refinements of Kadison [11] in the case of projectors in  $\mathcal{B}(\mathcal{H})$ , and the recent work [5].

The fact that  $\text{II}_1$  factors share many structural properties with the algebra of linear operators acting on  $\mathbb{C}^n$  makes them a natural context in which to extend majorization. In [5], Arveson and Kadison posed a (strong) version of the Schur–Horn theorem for  $\text{II}_1$  factors as a problem and proved related results. As a first step toward settling the Arveson–Kadison problem, the authors have proven in [3] a weaker version, related with the point of view developed in [13]. In this note we obtain a weak contractive version of submajorization within  $\text{II}_1$  factors in the spirit of [3] (Theorem 3.4). We also obtain an equivalent reformulation of the Arveson–Kadison problem (Theorem 4.1) using a characterization of spectral dominance and submajorization (Proposition 3.1).

## 2. Preliminaries

Throughout the paper  $\mathcal{M}$  denotes a  $\text{II}_1$  factor with normalized faithful normal trace  $\tau$ . We denote by  $\mathcal{M}^{\text{sa}}$ ,  $\mathcal{M}^+$ ,  $\mathcal{U}_{\mathcal{M}}$ , the sets of selfadjoint, positive, and unitary elements of  $\mathcal{M}$ . Given  $a \in \mathcal{M}^{\text{sa}}$  we denote its spectral measure by  $p^a$ . The characteristic function of the set  $\Delta$  is denoted by  $1_{\Delta}$ . We denote integration with respect to Lebesgue measure by  $dt$ .

Besides the usual operator norm in  $\mathcal{M}$ , we consider the Schatten norm induced by the trace,  $\|x\|_1 = \tau(|x|)$ . As we will be always dealing with bounded sets in a  $\text{II}_1$  factor, we can profit from the fact that the topology induced by the Schatten norm agrees with the  $\sigma$ -strong operator topology. Because of this we will express our results in terms of  $\sigma$ -strong closures although our computations are based on estimates for the Schatten norm. For  $X \subset \mathcal{M}$ , we shall denote by  $\overline{X}$  and  $\overline{X}^{\sigma\text{-sot}}$  the respective closures in the norm topology and in the  $\sigma$ -strong operator topology. For any set  $K$ ,  $\text{co } K$  denotes its convex hull.

### 2.1. Spectral scale and spectral preorders

The *spectral scale* [14] of  $a \in \mathcal{M}^{\text{sa}}$  is defined by

$$\lambda_a(t) = \min \{s \in \mathbb{R}: \tau(p^a(s, \infty)) \leq t\}, \quad t \in [0, 1].$$

The function  $\lambda_a: [0, 1] \rightarrow \mathbb{R}$  is non-increasing and right-continuous. The map  $a \mapsto \lambda_a$  is continuous both with respect to  $\|\cdot\|$  and  $\|\cdot\|_1$ , since [14]

$$\|\lambda_a - \lambda_b\|_{\infty} \leq \|a - b\|, \quad \|\lambda_a - \lambda_b\|_1 \leq \|a - b\|_1, \quad a, b \in \mathcal{M}^{\text{sa}}, \quad (1)$$

where the norms on the left are those of  $L^{\infty}([0, 1], dt)$  and  $L^1([0, 1], dt)$ , respectively. A useful property of the spectral scale is that we can use it to recover the trace, in the following sense:

$$\tau(a) = \int_0^1 \lambda_a(t) dt. \quad (2)$$

The unitary orbit of  $a \in \mathcal{M}^{\text{sa}}$  is the set  $\mathcal{U}_{\mathcal{M}}(a) = \{u^*au: u \in \mathcal{U}_{\mathcal{M}}\}$ . It is straightforward from the definition of the spectral scale that if  $b \in \mathcal{U}_{\mathcal{M}}(a)$ , then  $\lambda_a = \lambda_b$ . By the continuity (1),  $\lambda_b = \lambda_a$  for any  $b$  in the  $\|\cdot\|_1$ -closure or the  $\|\cdot\|$ -closure of the unitary orbit of  $a \in \mathcal{M}^{\text{sa}}$ . A converse of this fact was proven by Kamei. We summarize this information for future reference:

**Theorem 2.1.** ([9]) If  $a \in \mathcal{M}^{\text{sa}}$ , then

$$\overline{\mathcal{U}_{\mathcal{M}}(a)} = \overline{\mathcal{U}_{\mathcal{M}}(a)^{\sigma\text{-sot}}} = \{b \in \mathcal{M}^{\text{sa}}: \lambda_a = \lambda_b\}.$$

Let  $a, b \in \mathcal{M}^{\text{sa}}$ . We say that  $a$  is *spectrally dominated* by  $b$ , written  $a \precsim b$ , if any of the following (equivalent) statements holds:

- (i)  $\lambda_a(t) \leq \lambda_b(t)$ , for all  $t \in [0, 1]$ .
- (ii)  $\tau(p^a(t, \infty)) \leq \tau(p^b(t, \infty))$ , for all  $t$ .

We say that  $a$  is *submajorized* by  $b$ , written  $a \prec_w b$ , if

$$\int_0^s \lambda_a(t) dt \leq \int_0^s \lambda_b(t) dt, \quad \text{for every } s \in [0, 1].$$

If in addition  $\tau(a) = \tau(b)$ , then we say that  $a$  is *majorized* by  $b$ , written  $a \prec b$ .

**Remark 2.2.** Let  $a, b \in \mathcal{M}^{\text{sa}}$ . It is known [14] that

- (i) if  $a \leq b$ , then  $a \precsim b$ . Thus, using this and (2),

$$a \leq b \Rightarrow a \precsim b \Rightarrow a \prec_w b \Rightarrow \tau(a) \leq \tau(b);$$

- (ii) if  $v \in \mathcal{M}$  is a contraction ( $\|v\| \leq 1$ ), then  $v^*av \precsim a$ .

If  $\mathcal{N} \subset \mathcal{M}$  is a von Neumann subalgebra and  $b \in \mathcal{M}^{\text{sa}}$ , we denote by  $\Omega_{\mathcal{N}}(b)$  and  $\Theta_{\mathcal{N}}(b)$  the sets of elements in  $\mathcal{N}^{\text{sa}}$  that are, respectively, majorized and submajorized by  $b$ , i.e.,

$$\Omega_{\mathcal{N}}(b) = \{a \in \mathcal{N}^{\text{sa}}: a \prec b\}, \quad \Theta_{\mathcal{N}}(b) = \{a \in \mathcal{N}^{\text{sa}}: a \prec_w b\}.$$

The following result was proven in [3].

**Proposition 2.3.** Let  $b \in \mathcal{B}^{\text{sa}}$ , where  $\mathcal{B} \subset \mathcal{M}$  is a diffuse abelian von Neumann subalgebra. Then there exists a spectral resolution  $\{e(t)\}_{t \in [0, 1]} \subset \mathcal{B}$  with  $\tau(e(t)) = t$ , for every  $t \in [0, 1]$ , and such that

$$b = \int_0^1 \lambda_b(t) de(t).$$

The classical Schur–Horn theorem states that if  $\mathcal{N}$  is a type  $\text{I}_n$  factor,  $\mathcal{D} \subset \mathcal{N}$  is a masa,  $E_{\mathcal{D}}$  is the canonical projection onto  $\mathcal{D}$ , and  $b \in \mathcal{N}^{\text{sa}}$ , then

$$E_{\mathcal{D}}(\mathcal{U}_{\mathcal{N}}(b)) = \Omega_{\mathcal{D}}(b).$$

In [3], the authors proved the following related result for  $\text{II}_1$  factors.

**Theorem 2.4.** Let  $\mathcal{A} \subset \mathcal{M}$  be a diffuse abelian von Neumann subalgebra and let  $b \in \mathcal{M}^{\text{sa}}$ . Then

$$\overline{E_{\mathcal{A}}(\mathcal{U}_{\mathcal{M}}(b))^{\sigma\text{-sot}}} = \Omega_{\mathcal{A}}(b). \quad (3)$$

### 3. A contractive version of the Schur–Horn theorem

Given  $x \in \mathcal{M}$  we shall consider its contractive orbit  $\mathcal{C}_{\mathcal{M}}(x)$ , namely

$$\mathcal{C}_{\mathcal{M}}(x) := \{v^*xv : v \in \mathcal{M}, \|v\| \leq 1\}.$$

Using the results quoted in Section 2, we prove the following characterization of submajorization and spectral dominance.

**Proposition 3.1.** *Let  $a, b \in \mathcal{M}^+$ . Then*

- (i)  $a \prec_w b$  if and only if there exists  $c \in \mathcal{M}^+$  such that  $a \prec c \leq b$ . Moreover, if  $\mathcal{B} \subset \mathcal{M}$  is a diffuse abelian von Neumann subalgebra such that  $b \in \mathcal{B}^+$ , we can choose  $c \in \mathcal{B}^+$ .
- (ii)  $a \precsim b$  if and only if  $a \in \overline{\mathcal{C}_{\mathcal{M}}(b)}$ .

**Proof.** (i) Assume first that  $a \prec_w b$  and, without loss of generality, assume that  $b \in \mathcal{B}$  for a diffuse abelian subalgebra  $\mathcal{B} \subseteq \mathcal{M}$ . Let  $\{e(t)\}_{t \in [0,1]} \subseteq \mathcal{B}$  be a spectral resolution as in Proposition 2.3. Since the function  $g(s) := \int_0^s \lambda_b(t) dt$  is continuous and  $a \prec_w b$ , there exists  $s_0 \in [0, 1]$  such that  $\tau(a) = g(s_0)$ . Thus, if we let  $c = \int_0^{s_0} \lambda_b(t) de(t)$ , it is straightforward to verify that  $\lambda_c(t) = 1_{[0,s_0]} \lambda_b(t)$  for  $t \in [0, 1]$ . From this it follows that  $a \prec c$ . It is also clear that  $c \in \mathcal{B}$  and that  $c \leq b$ . Conversely, if there exists  $c \in \mathcal{M}^+$  such that  $a \prec c \leq b$ , then  $a \prec_w c$  and  $c \prec_w b$ , and so by transitivity we get  $a \prec_w b$ .

(ii) Let  $a, b \in \mathcal{M}^+$  with  $a \precsim b$ . Let  $\mathcal{B}$  be a diffuse abelian subalgebra with  $b \in \mathcal{B}$ , and let  $\{e(t)\}_{t \in [0,1]} \subseteq \mathcal{B}$  be as before. By hypothesis  $0 \leq \lambda_a \leq \lambda_b$ , so in particular  $\{\lambda_b = 0\} \subseteq \{\lambda_a = 0\}$ . Thus the function  $f = 1_{\{\lambda_b \neq 0\}} \cdot \lambda_a / \lambda_b$  is well defined,  $0 \leq f \leq 1$ , and  $f \cdot \lambda_b = \lambda_a$ . Therefore,  $v = \int_0^1 f(t)^{1/2} de(t) \in \mathcal{B}$  is a contraction such that

$$v^*bv = \int_0^1 \lambda_a(t) de(t) \quad \text{and thus} \quad \lambda_{v^*bv} = \lambda_a.$$

By Theorem 2.1 it follows that  $a \in \overline{\mathcal{U}_{\mathcal{M}}(v^*bv)} \subset \overline{\mathcal{C}_{\mathcal{M}}(b)}$ . To see the converse, let  $a \in \overline{\mathcal{C}_{\mathcal{M}}(b)}$ . Then  $a \precsim b$  since, by (ii) in Remark 2.2,  $v^*bv \precsim b$  for any contraction  $v \in \mathcal{M}$ , and by (1) the spectral scale is uniformly continuous with respect to the operator norm.  $\square$

In [6, Theorem 3.1], Hiai shows that, for  $b \in \mathcal{M}^+$ ,  $\{a \in \mathcal{M}^+ : a \precsim b\} = \overline{\mathcal{C}_{\mathcal{M}}(b)}^{\sigma\text{-sot}}$ . So, from Proposition 3.1, we obtain

**Corollary 3.2.** *If  $b \in \mathcal{M}^+$ , then  $\overline{\mathcal{C}_{\mathcal{M}}(b)}^{\sigma\text{-sot}} = \overline{\mathcal{C}_{\mathcal{M}}(b)}$ .*

**Lemma 3.3.** *Let  $\mathcal{N} \subset \mathcal{M}$  be a von Neumann subalgebra and let  $E_{\mathcal{N}}$  be the trace preserving conditional expectation onto  $\mathcal{N}$ . Then, for any  $b \in \mathcal{M}^+$ ,*

- (i)  $\|E_{\mathcal{N}}(b)\|_1 \leq \|b\|_1$ .
- (ii)  $\overline{E_{\mathcal{N}}(\mathcal{C}_{\mathcal{M}}(b))}^{\sigma\text{-sot}} \subset \Theta_{\mathcal{N}}(b) \cap \mathcal{N}^+$ .

**Proof.** (i) is proved in [3]. To see (ii) note that by Remark 2.2, for every  $v \in \mathcal{M}$  such that  $\|v\| \leq 1$ ,  $v^*bv \precsim b$ ; by Theorem 2.2 in [3],  $E_{\mathcal{N}}(v^*bv) \prec v^*bv$ . So by transitivity  $E_{\mathcal{N}}(v^*bv) \in$

$\Theta_{\mathcal{N}}(b) \cap \mathcal{N}^+$ . If  $(a_n)_{n \in \mathbb{N}} \subset E_{\mathcal{N}}(\mathcal{C}_{\mathcal{M}}(b))$  is such that  $\lim_{n \rightarrow \infty} \|a_n - a\|_1 = 0$  for some  $a \in \mathcal{N}$ , then necessarily  $a \in \mathcal{N}^+$ . By the previous argument we have that  $a_n \prec_w b$  for every  $n$ . Therefore, by (1),

$$\int_0^s \lambda_a(t) dt = \lim_{n \rightarrow \infty} \int_0^s \lambda_{a_n}(t) dt \leq \int_0^s \lambda_b(t) dt$$

and so  $a \prec_w b$ .  $\square$

Next we prove our main result, which complements Theorem 2.4 in the case of submajorization and contractive orbits.

**Theorem 3.4.** *Let  $\mathcal{A} \subset \mathcal{M}$  be a diffuse abelian von Neumann subalgebra of  $\mathcal{M}$  and let  $b \in \mathcal{M}^+$ . Then*

$$\overline{E_{\mathcal{A}}(\mathcal{C}_{\mathcal{M}}(b))}^{\sigma\text{-sot}} = \Theta_{\mathcal{A}}(b) \cap \mathcal{A}^+. \quad (4)$$

**Proof.** By (ii) in Lemma 3.3,  $\overline{E_{\mathcal{A}}(\mathcal{C}_{\mathcal{M}}(b))}^{\sigma\text{-sot}} \subset \Theta_{\mathcal{A}}(b) \cap \mathcal{A}^+$ . To prove the other inclusion, let  $a \in \mathcal{A}^+$  be such that  $a \prec_w b$ . By (i) in Proposition 3.1 there exists  $c \in \mathcal{M}^+$  such that  $a \prec c \leq b$ . By Theorem 2.4,

$$a \in \overline{E_{\mathcal{A}}(\mathcal{U}_{\mathcal{M}}(c))}^{\sigma\text{-sot}}. \quad (5)$$

Note that, since  $c \leq b$ ,  $c \preceq b$  (Remark 2.2). Thus, by (ii) in Proposition 3.1,

$$c \in \overline{\mathcal{C}_{\mathcal{M}}(b)}. \quad (6)$$

Let  $\varepsilon > 0$ . By (5) and (6) there exist  $u \in \mathcal{U}_{\mathcal{M}}$  and a contraction  $v \in \mathcal{M}$  such that  $\|a - E_{\mathcal{A}}(u^*cu)\|_1 \leq \varepsilon$  and  $\|c - v^*bv\| \leq \varepsilon$ . Therefore,

$$\|E_{\mathcal{A}}(u^*cu) - E_{\mathcal{A}}((vu)^*b(vu))\|_1 = \|E_{\mathcal{A}}(u^*(c - v^*bv)u)\|_1 \leq \varepsilon,$$

since  $\|x\|_1 \leq \|x\|$  and  $E_{\mathcal{A}} \circ Ad_u$  is a  $\|\cdot\|_1$ -contraction (Lemma 3.3). Thus

$$\begin{aligned} \|a - E_{\mathcal{A}}((vu)^*b(vu))\|_1 &\leq \|a - E_{\mathcal{A}}(u^*cu)\|_1 \\ &\quad + \|E_{\mathcal{A}}(u^*cu) - E_{\mathcal{A}}((vu)^*b(vu))\|_1 \\ &\leq 2\varepsilon. \end{aligned}$$

As  $\varepsilon$  was arbitrary, we get  $a \in \overline{E_{\mathcal{A}}(\mathcal{C}_{\mathcal{M}}(b))}^{\sigma\text{-sot}}$ , as desired.  $\square$

**Corollary 3.5.** *For each  $b \in \mathcal{M}^+$ , the set  $\overline{E_{\mathcal{A}}(\mathcal{C}_{\mathcal{M}}(b))}^{\sigma\text{-sot}}$  is convex and  $\sigma$ -weakly compact.*

**Proof.** By (3) in Theorem 2.5 of [6],

$$\Theta_{\mathcal{M}}(b) = \overline{\text{co}(\mathcal{C}_{\mathcal{M}}(b))}^{\sigma\text{-sot}}. \quad (7)$$

The right-hand side is bounded, convex, and  $\sigma$ -strongly closed, so it is  $\sigma$ -weakly closed and thus compact. Then

$$\Theta_{\mathcal{A}}(b) \cap \mathcal{A}^+ = \Theta_{\mathcal{M}}(b) \cap \mathcal{A}^+ = \overline{\text{co}(\mathcal{C}_{\mathcal{M}}(b))}^{\sigma\text{-sot}} \cap \mathcal{A}^+$$

is convex and  $\sigma$ -weakly compact. By Theorem 3.4, we are done.  $\square$

**Remark 3.6.** For any  $b \in \mathcal{M}^+$ , the property of  $\overline{E_{\mathcal{A}}(\mathcal{C}_{\mathcal{M}}(b))^{\sigma\text{-sot}}}$  being convex is essentially equivalent to Theorem 3.4. Indeed, assuming  $\overline{E_{\mathcal{A}}(\mathcal{C}_{\mathcal{M}}(b))^{\sigma\text{-sot}}}$  to be convex and using (7),

$$\begin{aligned}\Theta_{\mathcal{A}}(b) \cap \mathcal{A}^+ &\subset E_{\mathcal{A}}(\overline{\text{co}(\mathcal{C}_{\mathcal{M}}(b))^{\sigma\text{-sot}}}) \subset \overline{E_{\mathcal{A}}(\text{co}(\mathcal{C}_{\mathcal{M}}(b)))^{\sigma\text{-sot}}} \\ &= \overline{\text{co } E_{\mathcal{A}}((\mathcal{C}_{\mathcal{M}}(b)))^{\sigma\text{-sot}}} = \overline{E_{\mathcal{A}}(\mathcal{C}_{\mathcal{M}}(b))^{\sigma\text{-sot}}},\end{aligned}$$

where we have used that  $E_{\mathcal{A}}$  is  $\|\cdot\|_1$ -continuous (by (i) in Lemma 3.3). The reverse inclusion is given by (ii) in Lemma 3.3.

#### 4. A reformulation of the Arveson–Kadison problem

Let  $\mathcal{A} \subset \mathcal{M}$  be a masa and  $b \in \mathcal{M}^+$ . In [5], Arveson and Kadison pose the problem of whether

$$E_{\mathcal{A}}(\overline{\mathcal{U}_{\mathcal{M}}(b)}) = \Omega_{\mathcal{A}}(b). \quad (8)$$

Similarly, with regard to the results of the present paper, it is natural to ask whether

$$E_{\mathcal{A}}(\overline{\mathcal{C}_{\mathcal{M}}(b)}) = \Theta_{\mathcal{A}}(b) \cap \mathcal{A}^+. \quad (9)$$

It turns out that the two problems are equivalent, even in the broader class of diffuse abelian subalgebras.

**Theorem 4.1.** *Let  $\mathcal{A} \subset \mathcal{M}$  be a diffuse abelian subalgebra. Then the following statements are equivalent:*

- (i)  $\forall b \in \mathcal{M}^{\text{sa}}, E_{\mathcal{A}}(\overline{\mathcal{U}_{\mathcal{M}}(b)}) = \Omega_{\mathcal{A}}(b)$ ;
- (ii)  $\forall b \in \mathcal{M}^+, E_{\mathcal{A}}(\overline{\mathcal{C}_{\mathcal{M}}(b)}) = \Theta_{\mathcal{A}}(b) \cap \mathcal{A}^+$ .

**Proof.** Using arguments similar to those in Lemma 3.3 we can prove that for  $b \in \mathcal{M}^{\text{sa}}$ ,  $E_{\mathcal{A}}(\overline{\mathcal{U}_{\mathcal{M}}(b)}) \subset \Omega_{\mathcal{A}}(b)$ . If  $b \in \mathcal{M}^+$ , using the norm-continuity of  $E_{\mathcal{A}}$  and Lemma 3.3,

$$E_{\mathcal{A}}(\overline{\mathcal{C}_{\mathcal{M}}(b)}) \subset \overline{E_{\mathcal{A}}(\mathcal{C}_{\mathcal{M}}(b))} \subset \overline{E_{\mathcal{A}}(\mathcal{C}_{\mathcal{M}}(b))^{\sigma\text{-sot}}} \subset \Theta_{\mathcal{A}}(b) \cap \mathcal{A}^+.$$

(i)  $\Rightarrow$  (ii). Let  $a \in \Theta_{\mathcal{A}}(b) \cap \mathcal{A}^+$ ,  $b \in \mathcal{M}^+$ . Since  $\mathcal{A}$  is diffuse and abelian, by Proposition 2.3 there exists a spectral resolution of the identity  $\{e(t)\}_{t \in [0,1]} \subseteq \mathcal{A}$  such that  $\tau(e(t)) = t$  for  $t \in [0, 1]$  and such that  $a = \int_0^1 \lambda_a(t) de(t)$ . Consider the operator  $b' = \int_0^1 \lambda_b(t) de(t)$ . It is straightforward to verify that  $\lambda_{b'} = \lambda_b$  so that, by Theorem 2.1,  $\overline{\mathcal{U}_{\mathcal{M}}(b)} = \overline{\mathcal{U}_{\mathcal{M}}(b')}$ . From this last fact it follows that  $\overline{\mathcal{C}_{\mathcal{M}}(b)} = \overline{\mathcal{C}_{\mathcal{M}}(b')}$ , and so after replacing  $b$  by  $b'$  we can assume that  $b \in \mathcal{A}$ . By Proposition 3.1 there exists  $c \in \mathcal{A}^+$  such that  $a \prec c \leq b$  and by hypothesis we get  $a \in E_{\mathcal{A}}(\overline{\mathcal{U}_{\mathcal{M}}(c)})$ . Again by Proposition 3.1, since  $c \leq b$  implies  $c \preceq b$ , we get  $c \in \overline{\mathcal{C}_{\mathcal{M}}(b)}$ . Then  $\mathcal{U}_{\mathcal{M}}(c) \subset \overline{\mathcal{C}_{\mathcal{M}}(b)}$ , so we have  $a \in E_{\mathcal{A}}(\overline{\mathcal{C}_{\mathcal{M}}(b)})$ .

(ii)  $\Rightarrow$  (i). Let  $b \in \mathcal{M}^{\text{sa}}$ ,  $a \in \Omega_{\mathcal{A}}(b)$ . Since  $\lambda_{b+\alpha I} = \lambda_b + \alpha$ , then  $a \prec b$  if and only if  $a + \alpha \prec b + \alpha$ . Hence,  $\Omega_{\mathcal{A}}(b + \alpha I) = \Omega_{\mathcal{A}}(b) + \alpha I$ , and it is clear that  $E_{\mathcal{A}}(\overline{\mathcal{U}_{\mathcal{M}}(b + \alpha I)}) = E_{\mathcal{A}}(\overline{\mathcal{U}_{\mathcal{M}}(b)}) + \alpha I$ . Thus, we can assume without loss of generality that  $a, b \in \mathcal{M}^+$ . The following argument was inspired by the proof of Theorem 4.1 in [5]. Since in particular  $a \in \Theta_{\mathcal{A}}(b) \cap \mathcal{A}^+$ , by hypothesis there exist  $c \in \mathcal{M}^+$  and a sequence  $(v_n)_{n \in \mathbb{N}} \subset \mathcal{M}$  with  $\|v_n\| \leq 1$ ,  $n \in \mathbb{N}$ , such that

$$\lim_{n \rightarrow \infty} \|v_n^* b v_n - c\| = 0 \quad \text{and} \quad E_{\mathcal{A}}(c) = a. \quad (10)$$

So  $\tau(c) = \tau(a) = \tau(b)$ . Let  $p = p^b(0, \|b\|]$ .

**Claim.**  $\lim_{n \rightarrow \infty} \|p - |v_n^* p|\|_1 = 0$ .

Since  $\mathcal{M}$  is a finite factor, the partial isometry in the polar decomposition of  $v_n^* p$  can be extended to a unitary  $u_n$ : so  $v_n^* p = u_n |v_n^* p|$ . Thus

$$\begin{aligned} \|v_n^* b v_n - u_n b u_n^*\|_1 &= \|v_n^* p b (v_n^* p)^* - u_n b u_n^*\|_1 \\ &= \|u_n |v_n^* p| b |v_n^* p| u_n^* - u_n b u_n^*\|_1 \\ &= \| |v_n^* p| b |v_n^* p| - b \|_1 \\ &\leq \|(|v_n^* p| - p) b |v_n^* p|\|_1 + \|b (|v_n^* p| - p)\|_1 \\ &\leq \| |v_n^* p| - p \|_1 \|b |v_n^* p|\|_1 + \|b\| \| |v_n^* p| - p \|_1 \\ &\leq 2 \|b\| \| |v_n^* p| - p \|_1 \xrightarrow{n} 0. \end{aligned}$$

By (10) and the inequalities above,  $\lim_n \|c - u_n b u_n^*\|_1 = 0$ , and so  $c \in \overline{\mathcal{U}_{\mathcal{M}}(b)}^{\sigma\text{-sot}}$ . Using Theorem 2.1,

$$a = E_{\mathcal{A}}(c) \in E_{\mathcal{A}}(\overline{\mathcal{U}_{\mathcal{M}}(b)}^{\sigma\text{-sot}}) = E_{\mathcal{A}}(\overline{\mathcal{U}_{\mathcal{M}}(b)}).$$

**Proof of the claim.** Since  $\|v_n\| \leq 1$ ,  $p v_n v_n^* p \leq p$ , and so  $|v_n^* p| \leq p$ . Then by (10),

$$\begin{aligned} 0 &\leq \lim_n \tau((1 - v_n v_n^*)b) = \lim_n \tau(b - v_n^* b v_n) \\ &= \lim_n \tau(c - v_n^* b v_n) \leq \lim_n \|c - v_n^* b v_n\| = 0. \end{aligned}$$

Let  $\varepsilon > 0$ . Since  $b(b + \delta)^{-1} \nearrow p$  strongly when  $\delta \rightarrow 0$  and

$$0 \leq \tau((1 - v_n v_n^*)(p - b(b + \delta)^{-1})) \leq \tau(p - b(b + \delta)^{-1}),$$

we can choose  $\delta$  such that  $\tau((1 - v_n v_n^*)p) \leq \varepsilon + \tau((1 - v_n v_n^*)b(b + \delta)^{-1})$  for every  $n \in \mathbb{N}$ . Then, choosing  $n$  such that  $\tau((1 - v_n v_n^*)b) \leq \|(b + \delta)^{-1}\|^{-1} \varepsilon$ , we obtain

$$\begin{aligned} 0 &\leq \tau((1 - v_n v_n^*)p) \leq \varepsilon + \tau((1 - v_n v_n^*)b(b + \delta)^{-1}) \\ &= \varepsilon + \tau((1 - v_n v_n^*)^{1/2} b^{1/2} (b + \delta)^{-1} b^{1/2} (1 - v_n v_n^*)^{1/2}) \\ &\leq \varepsilon + \|(b + \delta)^{-1}\| \tau((1 - v_n v_n^*)b) \leq 2\varepsilon. \end{aligned}$$

Therefore,  $\lim_n \tau((1 - v_n v_n^*)p) = 0$ . For any  $x \in \mathcal{M}^+$  with  $\|x\| \leq 1$ ,  $x - x^2 = x(1 - x) = x^{1/2}(1 - x)x^{1/2} \geq 0$ . Since  $\|v_n^* p\| \leq 1$ , we conclude that  $|v_n^* p|^2 \leq |v_n^* p|$ , and so

$$\begin{aligned} \|p - |v_n^* p|\|_1 &= \tau(p - |v_n^* p|) \leq \tau(p - |v_n^* p|^2) \\ &= \tau(p - p v_n v_n^* p) = \tau((1 - v_n v_n^*)p) \rightarrow 0. \quad \square \end{aligned}$$

We finish with the following remark concerning the relation between our main result and the problem (9). The characterization in Theorem 3.4 of the positive operators in a diffuse abelian subalgebra  $\mathcal{A}$  majorized by a fixed  $b \in \mathcal{M}^+$  is weaker than that posed in (9), since in general (using Corollary 3.2)

$$E_{\mathcal{A}}(\overline{\mathcal{C}_{\mathcal{M}}(b)}) = E_{\mathcal{A}}(\overline{\mathcal{C}_{\mathcal{M}}(b)}^{\sigma\text{-sot}}) \subset \overline{E_{\mathcal{A}}(\mathcal{C}_{\mathcal{M}}(b))}^{\sigma\text{-sot}}. \quad (11)$$

By Theorems 3.4 and 4.1, an affirmative answer to the Arveson–Kadison problem would imply equality in (11) and, conversely, equality in (11) would settle the Arveson–Kadison problem affirmatively.

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## References

- [1] T. Ando, Majorization, Doubly Stochastic Matrices and Comparison of Eigenvalues, Hokkaido University Research Institute of Applied Electricity Division of Applied Mathematics, Sapporo, 1982.
- [2] J. Antezana, P. Massey, M. Ruiz, D. Stojanoff, The Schur–Horn theorem for operators and frames with prescribed norms and frame operator, *Illinois J. Math.*, in press, arXiv: math.FA/0508646.
- [3] M. Argerami, P. Massey, A Schur–Horn theorem in  $\text{II}_1$  factors, *Indiana Univ. J. Math.*, in press.
- [4] B.C. Arnold, Majorization and the Lorenz Order: A Brief Introduction, *Lecture Notes in Statist.*, vol. 43, Springer-Verlag, Berlin, 1987.
- [5] W. Arveson, R.V. Kadison, Diagonals of self-adjoint operators, in: D.R. Larson, D. Han, P.E.T. Jorgensen (Eds.), *Operator Theory, Operator Algebras, and Applications*, in: *Contemp. Math.*, vol. 414, Amer. Math. Soc., Providence, RI, 2006.
- [6] F. Hiai, Majorization and stochastic maps in von Neumann algebras, *J. Math. Anal. Appl.* 127 (1) (1987) 18–48.
- [7] F. Hiai, Spectral majorization between normal operators in von Neumann algebras, in: *Operator Algebras and Operator Theory*, Craiova, 1989, in: *Pitman Res. Notes Math. Ser.*, vol. 271, Longman Sci. Tech., Harlow, 1992, pp. 78–115.
- [8] A. Horn, Doubly stochastic matrices and the diagonal of a rotation matrix, *Amer. J. Math.* 76 (3) (1954) 620–630.
- [9] E. Kamei, Majorization in finite factors, *Math. Japonica* 28 (4) (1983) 495–499.
- [10] R. Kadison, The Pythagorean theorem: I. The finite case, *Proc. Natl. Acad. Sci. USA* 99 (7) (2002) 4178–4184.
- [11] R. Kadison, The Pythagorean theorem: II. The infinite discrete case, *Proc. Natl. Acad. Sci. USA* 99 (8) (2003) 5217–5222.
- [12] A.W. Marshall, I. Olkin, *Inequalities: Theory of Majorization and Its Applications*, *Math. Sci. Eng.*, vol. 143, Academic Press, New York, 1979.
- [13] A. Neumann, An infinite-dimensional version of the Schur–Horn convexity theorem, *J. Funct. Anal.* 161 (1999) 418–451.
- [14] D. Petz, Spectral scale of selfadjoint operators and trace inequalities, *J. Math. Anal. Appl.* 109 (1985) 74–82.
- [15] I. Schur, Über eine klasse von mittlebildungen mit anwendungen auf der determinantentheorie, *Sitzungsber. Berliner Mat. Ges.* 22 (1923) 9–29.